A Symbolic Computation-Based Method and Two Nonlinear Evolution Equations for Water Waves

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A symbolic-computation-based method, which has been newly proposed, is considered for a (2+1)-dimensional generalization of shallow water wave equations and a coupled set of the (2+1)-dimensional integrable dispersive long wave equations. New sets of soliton-like solutions are constructed, along with solitary waves.

1. (2+1)-dimensional Generalized Shallow Water Wave Equation

In order to exactly solve for a (2+1)-dimensional generalization of the shallow water wave equations [1, 2]

$$u_{yt} + u_{xxxy} - 3\,u_{xx}\,u_y - 3\,u_x\,u_{xy} = 0\,, \qquad (1)$$

we introduce a symbolic-computation-based method newly proposed [3, 4], so as to see that

$$u(x, y, t) = A\,\partial_x\,w[z(x, y, t)] + \Psi(x, y, t)\,, \qquad (2)$$

where A is a non-zero constant, and $\Psi(x, y, t)$, $z(x, y, t)$ and $w(z)$ are differentiable functions.

Then, the requirement is that the result be an ordinary differential equation for $w(z)$, and $z(x, y, t)$ be obtained correspondingly. We substitute (2) back into (1), and equate to zero the coefficients of the terms with the highest power of the differential coefficients of $z(x, y, t)$, i.e., of the $z_x^4\,z_y$ terms, so that

$$-6\,A w''\,w''' + w^{(5)} = 0 \;\rightarrow\; w(z) = -\frac{2}{A}\ln(z)\,. \qquad (3)$$

After a study on the vanishing coefficients of the w' terms, we try

$$\Psi(x, y, t) = \Psi(t)\,, \qquad (4)$$

and conjecture that

$$z(x, y, t) = 1 + e^{\gamma x + \Omega(y, t)}\,, \qquad (5)$$

where $\gamma \neq 0$ is a constant while $\Omega(y, t)$ is a differentiable function.

After symbolic manipulations for the expression of $\Omega(y, t)$, etc., we obtain a set of the soliton-like solutions for (1) as follows:

$$u(x, y, t) = -\gamma \cdot \tanh\left[\frac{\gamma\,x + \Theta(y) - \gamma^3\,t}{2}\right] + \Phi(t)\,, \qquad (6)$$

where $\Phi(t) = \Psi(t) - \gamma$ and $\Theta(y)$ are arbitrary differentiable functions. Solitary waves are a special case with $\Theta(y) = a\,y + b$ and $\Phi(t) = c$, where a, b and c are constants easy to be determined. This solution should be

2. (2+1)-dimensional Integrable Dispersive Long Wave Equations

The coupled set

$$u_{ty} = -\eta_{xx} - \tfrac{1}{2}(u^2)_{xy}\,, \qquad (7)$$

$$\eta_t = -(u\,\eta + u + u_{xy})_x \qquad (8)$$

has been used to model wide channels or open seas and was studied in [5–7], where $\eta(x, y, t)$ represents the amplitude of a surface wave which propagates in the (x, y) plane with the horizontal velocity $u(x, y, t)$.

Using the aforementioned method, we have done preliminary work aiming at some solutions of (7) and (8) [4]. Recently, we have performed extensive work, which is going to be published elsewhere, with sample solutions (including solitary waves). Here we briefly outline the new families of soliton-like solutions:

Family I:

$$u^{(I)}(x, y, t) = \theta \cdot \left\{\tanh\left[\frac{\theta \cdot x + \Psi(y, t) - \ln \Sigma(y, t)}{2}\right] + 1\right\}\,, \qquad (9)$$

$$\eta^{(I)}(x, y, t) = \frac{\theta}{2} \cdot \left[\Psi_y(y, t) - \frac{\Sigma_y(y, t)}{\Sigma(y, t)}\right] \cdot \operatorname{sech}^2\left[\frac{\theta \cdot x + \Psi(y, t) - \ln \Sigma(y, t)}{2}\right] - 1\,, \qquad (10)$$

where the differentiable functions $\Sigma(y, t) \neq 0$, $\Psi(y, t)$ and their derivatives, along with the constant $\theta \neq 0$, satisfy

$$\begin{cases} -\theta^2\,\Sigma^2\,\Psi_y - \Sigma^2\,\Psi_t\,\Psi_y + \Sigma\Sigma_t\,\Psi_y + \theta^2\,\Sigma\Sigma_y + \Sigma\,\Psi_t\,\Sigma_y - \Sigma_t\,\Sigma_y = 0, \\ \Sigma_t\,\Sigma_y + \Sigma^2\,\Psi_{yt} - \Sigma\Sigma_{yt} = 0. \end{cases} \qquad (11)$$

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Family II:

$$u^{(II)}(x,y,t)=\Theta(y) \cdot \left\{\tanh\left[\frac{\Theta(y)\cdot x+\Psi(y,t)-\ln\Sigma(y,t)}{2}\right]+1\right\}, \tag{12}$$

$$\begin{aligned}\eta^{(II)}(x,y,t)=&\frac{1}{2\cdot\Sigma(y,t)}\operatorname{sech}^2\left[\frac{\Theta(y)\cdot x+\Psi(y,t)-\ln\Sigma(y,t)}{2}\right]\\ &\cdot[e^{\Theta(y)\cdot x+\Psi(y,t)}\Theta_y(y)\\ &+\Sigma(y,t)\,\Theta_y(y)+x\cdot\Theta(y)\,\Sigma(y,t)\,\Theta_y(y)\\ &+\Theta(y)\,\Sigma(y,t)\,\Psi_y(y,t)-\Theta(y)\,\Sigma_y(y,t)]-1,\end{aligned} \tag{13}$$

where the differentiable functions $\Theta(y)\neq 0$, $\Sigma(y,t)\neq 0$, $\Psi(y,t)$ and their derivatives satisfy

$$\begin{cases}-\Theta^2\,\Sigma-\Sigma\,\Psi_t+\Sigma_t=0,\\ \Sigma^2(\Theta^2)_y+\Sigma_t\,\Sigma_y+\Sigma^2\,\Psi_{yt}-\Sigma\Sigma_{yt}=0.\end{cases} \tag{14}$$

Family III:

$$u^{(III)}(x,y,t)=\Theta(t)\cdot\left\{\tanh\left[\frac{\Theta(t)\cdot x-\ln\Phi(t)}{2}\right]+1\right\}, \tag{15}$$

$$\eta^{(III)}(x,y,t)=-1, \tag{16}$$

where $\Theta(t)\neq 0$ and $\Phi(t)\neq 0$ are arbitrary, differentiable functions. In this family, $u^{(III)}=u(x,t)$ is independent of y, and $\eta^{(III)}$ is a constant.

Those families give us new information on the motion of the surface waves on open seas and wide channels.

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